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Non-frustrated random spin systems with gauge symmetry

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Abstract. We introduce a class of random spin systems without frustration, which satisfies the conditions for gauge symmetry. The method of gauge transformation provides several properties on the phase diagram of gauge symmetric models; this method is almost rigorous, while the derivation for the absence of re-entrance contains unproved assumption. In the present random models, the absence of re-entrance can be shown exactly. Furthermore, the phase diagram and the critical properties are exactly related with those in the original pure system. The present random systems correspond to the Mattis model with asymmetric bond distribution in the Ising case, and a kind of the gauge glass model in the XY case. In the case of the clock model in two dimensions, we find a new thermodynamic phase which has long-range spin-glass correlation with critical (power-decaying) ferromagnetic correlation.

1. Introduction

It has been a fascinating subject in physics to study the effects of randomness on cooperative systems. One of the most interesting problems is the spin glass (SG) [1]. Since the pioneering work of Edwards and Anderson (EA) [2], the SG problem has been investigated as a phase transition in nature. Up to the mean-field model, which originates from the work of Sherrington and Kirkpatrick [3], the theoretical picture of the EA model has been established based on replica-symmetry breaking by Parisi [4]. The system exhibits a typical many-valley structure in the phase space, the mixed phase of the ferromagnetic (FM) and the SG phases, the Almeida–Thouless line in the external field and so on. Recent progress in this field is devoted to short-range systems to know the lower critical dimension of the SG transition and then to check the validity of the Parisi picture in real spin glasses. The SG transition has been observed numerically in three-dimensional EA Ising models [5–7], while it has been denied in the two-dimensional Ising and the three-dimensional Heisenberg models [8, 9].

In connection with granular superconductor, the XY gauge glass model

$$\mathcal{H} = -J \sum_{\langle ij\rangle} \cos(\phi_i - \phi_j + \omega_{ij}) \tag{1.1}$$

has attracted much attention, where ϕ_i , $\omega_{ij} \in [0, 2\pi)$. In this model, the existence of the SG ordering has been confirmed numerically in three dimensions [10–13] in contrast with $\pm J$ counterpart. There is a controversy about the existence of re-entrance in the weakly random regime. The real-space renormalization group method with the Coulomb gas description leads to the re-entrant transition in two and more dimensions [14, 15]. Experimental as well as Monte Carlo studies indicate no re-entrance in two dimensions [16–18]. A Migdal–Kadanoff type renormalization group calculation in three dimensions also failed to discover

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re-entrance [13, 19]. Recently, a modified analysis is presented for the Coulomb gas method and shows the absence of it [20].

The re-entrant SG phase observed in real experiments [1, 21] and obtained in the replica-symmetric solution in the mean field theory[3] does not appear in the solution by Parisi [4]. The method of gauge transformation [22–24] indicates the absence of re-entrant transition in short-range EA Ising models with the $\pm J$ or the Gaussian-bond distribution and in the XY gauge glass model. It is consistent with numerical results [25–27] as well as recent experiments in Ising-like spin glasses [28]. The method of gauge transformation is a powerful technique to derive exact properties of random spin systems. It was first applied to Ising spin glasses by Nishimori [22]. Kitatani [23] introduced the modified model providing a plausible argument for the absence of re-entrance. Ozeki and Nishimori [24] generalize the method to random systems with various symmetries including the XY gauge glass model.

Since randomness and frustration make it difficult to examine the short-range systems analytically as well as numerically, only a few things have been confirmed definitely in spite of a huge number of numerical studies. Thus, it is highly desirable to have solvable models for understanding the thermodynamic and critical behaviours in random spin systems. The asymmetric Mattis model is a random Ising model without frustrations. The phase diagram is obtained exactly [29, 30]. The critical exponents for both SG and FM ordering are related with those in the pure Ising model. This model is gauge symmetric, and the method of gauge transformation can be applied to it [30].

We propose a randomization procedure for general spin systems, which provides a class of random spin systems without frustration. These random systems are corresponding to the asymmetric Mattis model in the Ising case. All models have gauge symmetry, and the method of gauge transformation can be applied to them; the phase diagram is obtained exactly. The critical exponents for both SG and FM ordering are related with those in the original pure system. Since these models can be analysed exactly, they help us to understand the phase transition and the critical phenomena in random systems and to check the efficiency of methods to analyse them.

In section 2, a general representation of random spin systems is introduced to treat various systems coherently. Using this representation, we construct a class of non-frustrated random spin systems with gauge symmetry, and derive their exact properties in section 3. In section 4, phase diagrams and critical exponents are obtained for several specific cases. The last section is devoted to remarks.

2. General representation for random spin systems with gauge symmetry

In order to treat various spin systems coherently, first, we define the general representation for random spin systems with gauge symmetry introduced by Ozeki and Nishimori [24]. Let us consider classical spin systems on a lattice with N lattice points. We make no restrictions on the type or the dimension of the lattice, whereas one may suppose a usual d-dimensional hypercubic lattice. The variable ϕ_i represents a local spin-state at the ith lattice point. The set Φ is the space of spin states at one lattice point with a measure $\mathrm{d}\mu(\phi)$. We denote $\boldsymbol{\phi} \equiv (\phi_1, \ldots, \phi_N) \in \Phi^N$ a global spin-configuration for the total N-spin system. The measure of $\boldsymbol{\phi}$ is expressed as $\mathrm{d}\mu\{\boldsymbol{\phi}\} = \prod_{i=1}^{N} \mathrm{d}\mu(\phi_i)$. The non-random Hamiltonian is denoted by

$$\mathcal{H}_0(\boldsymbol{\phi}) = J\tilde{\mathcal{H}}_0(\boldsymbol{\phi}) \tag{2.1}$$

where $\tilde{\mathcal{H}}_0$ is the dimensionless part of $\mathcal{H}_0$. The notations are summarized in table 1.

Table 1. Definitions and notations of variables.

	Spin or thermal	Randomness
Variable	ϕ_i	ω_n
Set of space	Φ	Ω
Measure of variable	$\mathrm{d}\mu(\phi_i)$	$\mathrm{d}\nu(\omega_n)$
Number of variables	N	N_{R}
Global configuration	$\boldsymbol{\phi}$	$\boldsymbol{\omega}$
Measure of global Ccnfiguration	$\mathrm{d}\mu\{\boldsymbol{\phi}\}$	$\mathrm{d}\nu\{\boldsymbol{\omega}\}$
Local gauge transformation	U_ψ	V_ψ
Average	$\langle\cdots\rangle_K$	$[\cdots]_{\mathrm{c}}$
Control parameter	$K = J/k_{\mathrm{B}}T$	K_p
Distribution	$\exp\{-K\tilde{\mathcal{H}}(\boldsymbol{\phi};\boldsymbol{\omega})\}/Z(K;\boldsymbol{\omega})$	$P(\boldsymbol{\omega}; K_p)$

Table 2. Examples of models with symmetries and group operations.

Model	Set Φ	Operation $\phi \circ \psi$	Identity ϕ_{E}	Inverse $\bar{\phi}$
Ising	$\{+1, -1\}$	$\phi\psi$	1	ϕ
Z_q	$\{0, 1, \ldots q-1\}$	$\phi + \psi \pmod q$	0	$-\phi$
XY	$[0, 2\pi)$	$\phi + \psi \pmod{2\pi}$	0	$-\phi$
$SU(2)$	$\{SU(2)$ matrices$\}$	$\phi\psi$	I	$\phi^\dagger$

Based on the spin system introduced above, random systems are generally expressed in the following way. Consider N_{R} random variables $\omega_n \in \Omega$ $(n = 1, 2, \ldots, N_{\mathrm{R}})$. The set Ω is the space of randomness ω_n with a non-negative measure $\mathrm{d}\nu(\omega_n)$. We denote $\boldsymbol{\omega} \equiv (\omega_1, \ldots, \omega_{N_{\mathrm{R}}}) \in \Omega^{N_{\mathrm{R}}}$ a global random-configuration for the total N_{R}-random variables. The measure of $\boldsymbol{\omega}$ is expressed as $\mathrm{d}\nu\{\boldsymbol{\omega}\} = \prod_{n=1}^{N_{\mathrm{R}}} \mathrm{d}\nu(\omega_n)$. The Hamiltonian is a function of both $\boldsymbol{\phi}$ and $\boldsymbol{\omega}$;

$$\mathcal{H}(\boldsymbol{\phi};\boldsymbol{\omega}) = J\tilde{\mathcal{H}}(\boldsymbol{\phi};\boldsymbol{\omega}). \tag{2.2}$$

We assume that there exists a particular state of randomness, ω_{E}, with which the Hamiltonian $\mathcal{H}(\boldsymbol{\phi};\boldsymbol{\omega})$ becomes identical to $\mathcal{H}_0(\boldsymbol{\phi})$ if $\omega_n = \omega_{\mathrm{E}}$ for all n. The thermal average at temperature $T = J/k_{\mathrm{B}}K$ is expressed as

$$\langle\cdots\rangle_K \equiv Z(K,\boldsymbol{\omega})^{-1} \int \mathrm{d}\mu\{\boldsymbol{\phi}\} \ldots \exp\{-K\tilde{\mathcal{H}}(\boldsymbol{\phi};\boldsymbol{\omega})\} \tag{2.3}$$

where $Z(K;\boldsymbol{\omega})$ is the partition function for the configuration $\boldsymbol{\omega}$. The average over random configurations is defined by

$$[\cdots]_{\mathrm{c}} \equiv \int \mathrm{d}\nu\{\boldsymbol{\omega}\}\, P(\boldsymbol{\omega}; K_p) \ldots \tag{2.4}$$

where $P(\boldsymbol{\omega}; K_p)$ is the probability weight of the configuration $\boldsymbol{\omega}$. The non-negative parameter K_p controls the degree of randomness. Without loss of generality, we assume that the distribution function $P(\boldsymbol{\omega}; K_p)$ approaches to the delta function,

$$P(\boldsymbol{\omega}; K_p) \to \prod_n \delta(\omega_n, \omega_{\mathrm{E}}) \tag{2.5}$$

when $K_p \to \infty$, and becomes the most random one (the uniform distribution in many cases) when $K_p = 0$.

The method of gauge transformation is a powerful technique to derive exact properties of random spin systems. We summarize the applicability and the results in the appendix. If the system satisfies the conditions (I)–(V) in the appendix, the method can be applied to it. In order to make the system gauge symmetric, we assume that the set Φ forms a topological group with an operation denoted by $\phi \circ \psi$; the identity is ϕ_{E} and the inversion is represented by $\bar{\phi}$. The gauge transformation for spin variables associated with a spin configuration $\boldsymbol{\psi} = (\psi_1, \psi_2, \ldots, \psi_N)$ is defined as

$$U_\psi : \phi_i \to \phi_i' = \phi_i \circ \bar{\psi}_i \qquad \text{for all } i \tag{2.6}$$

where $\psi_i \in \Phi$. Since Φ is a topological group, the transformation U_ψ forms a group homomorphic to Φ^N and the measure $\mathrm{d}\mu\{\phi\}$ can be chosen as an invariant Haar measure;

$$\int \mathrm{d}\mu\{\phi\} \ldots = \int \mathrm{d}\mu\{\phi\}\, U_\psi \ldots . \tag{2.7}$$

We also assume that there exists a gauge transformation for random variables associated with $\boldsymbol{\psi} \in \Phi^N$;

$$V_\psi : \omega_n \to \omega_n' \in \Omega \qquad \text{for all } n \tag{2.8}$$

which forms a group homomorphic to Φ^N. Then, one can choose the measure $\mathrm{d}\nu\{\boldsymbol{\omega}\}$ as an invariant measure;

$$\int \mathrm{d}\nu\{\boldsymbol{\omega}\} V_\psi \ldots = \int \mathrm{d}\nu\{\boldsymbol{\omega}\} \ldots . \tag{2.9}$$

Usually we set the element ω_{E} the identity of the group.

A global gauge transformation associated with a spin-state $\psi \in \Phi$ is defined by

$$\tilde{U}_\psi : \phi_i \to \psi \circ \phi_i \qquad \text{for all } i. \tag{2.10}$$

We assume the invariance of the Hamiltonian in terms of the global transformation,

$$\tilde{U}_\psi \mathcal{H}(\phi; \boldsymbol{\omega}) = \mathcal{H}(\phi; \boldsymbol{\omega}) \tag{2.11}$$

so that at least one phase transition takes place with a kind of symmetry breaking. Let $\gamma(\phi)$ be an irreducible representation of Φ, which, in general, is a matrix in non-Abelian cases. For simplicity, we restrict the symmetry-only Abelian cases hereafter, which means that $\gamma(\phi)$ is a number. In the paramagnetic (PM) phase or in finite systems, the symmetry

$$\langle \gamma(\phi_i) \rangle_K = 0 \tag{2.12}$$

can be easily seen from equations (2.7) and (2.11). This symmetry is supposed to be broken in the ordered phase. We treat only FM pure systems involving the FM as well as the Kosterlitz–Thouless (KT) phases [31]. Generalization is simple and straightforward; the FM and the SG correlation functions are defined as

$$f(r; K, K_p) \equiv [\langle \gamma(\bar{\phi}_0 \circ \phi_r) \rangle_K]_c \tag{2.13}$$

$$g(r; K, K_p) \equiv [\langle \gamma(\bar{\phi}_0 \circ \phi_r) \rangle_K \langle \gamma^\dagger(\bar{\phi}_0 \circ \phi_r) \rangle_K]_c. \tag{2.14}$$

The FM and the SG order parameters,

$$m(K, K_p)^2 \equiv \lim_{r\to\infty} f(r; K, K_p) \tag{2.15}$$

$$q(K, K_p)^2 \equiv \lim_{r\to\infty} g(r; K, K_p) \tag{2.16}$$

distinguish the phase at (K, K_p) in such a way as

$m = 0$ and $q = 0$ in the PM phase,
$m > 0$ and $q > 0$ in the FM phase,
$m = 0$ and $q > 0$ in the SG phase.

3. Non-frustrated random spin models

Following the asymmetric Mattis model [30] in the Ising case, we introduce non-frustrated random spin systems for various symmetries satisfying the conditions (I)–(V) in the appendix. The random variable ω_i is assigned to each lattice site, and is an element of the same group with spin variables, $\omega_i \in \Phi$. The measure for $\boldsymbol{\omega}$ is identical with that for $\boldsymbol{\phi}$; $\mathrm{d}\nu(\boldsymbol{\omega}) = \mathrm{d}\mu(\boldsymbol{\omega})$. The Hamiltonian is expressed by use of that in the pure system as

$$\mathcal{H}(\boldsymbol{\phi}; \boldsymbol{\omega}) = \mathcal{H}_0(\boldsymbol{\phi} \circ \boldsymbol{\omega}) \tag{3.1}$$

where $\boldsymbol{\phi} \circ \boldsymbol{\omega} = (\phi_1 \circ \omega_1, \phi_2 \circ \omega_2, \ldots, \phi_N \circ \omega_N)$. The distribution of randomness has the same form with the thermal distribution;

$$P(\boldsymbol{\omega}; K_p) = \exp\{-K_p \tilde{\mathcal{H}}_0(\boldsymbol{\omega})\}/Z_0(K_p). \tag{3.2}$$

We regard this system as non-frustrated in the following sense. Suppose that the spin-configuration $\boldsymbol{\phi}_{\mathrm{g}}$ is one of the ground state(s) in the pure system, which means

$$E_{\mathrm{g}} \equiv \min_{\boldsymbol{\phi} \in \Phi^N} \mathcal{H}_0(\boldsymbol{\phi}) = \mathcal{H}_0(\boldsymbol{\phi}_{\mathrm{g}}). \tag{3.3}$$

In the random system at any random-configuration $\boldsymbol{\omega}$, one can find a spin-configuration, $\boldsymbol{\phi}_{\mathrm{g}} \circ \bar{\boldsymbol{\omega}} = (\phi_{\mathrm{g}1} \circ \bar{\omega}_1, \phi_{\mathrm{g}2} \circ \bar{\omega}_2, \ldots, \phi_{\mathrm{g}N} \circ \bar{\omega}_N)$, which has the same energy as E_{g};

$$\mathcal{H}(\boldsymbol{\phi}_{\mathrm{g}} \circ \bar{\boldsymbol{\omega}}; \boldsymbol{\omega}) = E_{\mathrm{g}}. \tag{3.4}$$

Therefore, the ground-state energy of the random system is always the same as that of the pure system. This indicates the system non-frustrated. In the above argument, since we do not restrict the type of the lattice, the usual plaquette notion cannot be used, which is necessary for 'non-frustrated' in usual cases such as systems with nearest-neighbour interactions on hypercubic lattices. The present randomization guarantees the system non-frustrated only when the original pure system is non-frustrated.

As an example, the pure Ising model,

$$\mathcal{H}_0(\boldsymbol{\phi}) = -J \sum_{\langle ij \rangle} \phi_i \phi_j \qquad \phi_i = \pm 1 \tag{3.5}$$

provides the asymmetric Mattis model [30],

$$\mathcal{H}(\boldsymbol{\phi}; \boldsymbol{\omega}) = -J \sum_{\langle ij \rangle} \omega_i \omega_j \phi_i \phi_j \tag{3.6}$$

with

$$P(\boldsymbol{\omega}; K_p) \propto \exp\left\{K_p \sum_{\langle ij \rangle} \omega_i \omega_j\right\} \tag{3.7}$$

$$U_\psi : \phi_i \to \phi_i \psi_i. \tag{3.8}$$

Another example is the XY model,

$$\mathcal{H}_0(\boldsymbol{\phi}) = -J \sum_{\langle ij \rangle} \cos(\phi_i - \phi_j) \qquad \phi_i \in [0, 2\pi) \tag{3.9}$$

providing a random model

$$\mathcal{H}(\boldsymbol{\phi}; \boldsymbol{\omega}) = -J \sum_{\langle ij \rangle} \cos(\phi_i - \phi_j + \omega_i - \omega_j) \tag{3.10}$$

with

$$P(\boldsymbol{\omega}; K_p) \propto \exp\left\{K_p \sum_{\langle ij \rangle} \cos(\omega_i - \omega_j)\right\} \tag{3.11}$$

$$U_\psi : \phi_i \to \phi_i - \psi_i. \tag{3.12}$$

The Hamiltonian (3.9) is similar to that of the XY gauge glass (1.1), while the randomness is different. If each spin variable takes only q discrete values,

$$\phi_i \in \left\{0, \frac{2\pi}{q}, \frac{4\pi}{q}, \ldots, \frac{2(q-1)\pi}{q}\right\} \equiv \Phi_q \tag{3.13}$$

the pure Hamiltonian (3.9) describes the q-state clock model and provides a random model whose Hamiltonian and the probability distribution are the same as (3.10) and (3.11) with discrete random variables $\omega_{ij} \in \Phi_q$. This model has Z_q symmetry instead of $O(2)$. It is also possible to apply the present randomization to other pure models; the q-state Potts model provides

$$\mathcal{H}(\phi;\boldsymbol{\omega}) = -J\sum_{\langle ij\rangle}\delta[\phi_i - \phi_j + \omega_i - \omega_j, 0] \qquad \phi_i, \omega_i \in \Phi_q \tag{3.14}$$

$$P(\boldsymbol{\omega}; K_p) \propto \exp\left\{K_p \sum_{\langle ij\rangle}\delta[\omega_i - \omega_j, 0]\right\} \tag{3.15}$$

and the Ising model with four-spin interactions provides

$$\mathcal{H}(\phi;\boldsymbol{\omega}) = -J\sum_{\langle ijkl\rangle}\omega_i\omega_j\omega_k\omega_l\phi_i\phi_j\phi_k\phi_l \tag{3.16}$$

$$P(\boldsymbol{\omega}; K_p) \propto \exp\left\{K_p \sum_{\langle ijkl\rangle}\omega_i\omega_j\omega_k\omega_l\right\}. \tag{3.17}$$

The behaviour of these random models are categorized and summarized in the next section.

Since Ω forms the same group as Φ, the gauge transformation for $\boldsymbol{\omega}$ is identical with that for ϕ;

$$V_\psi : \omega_i \to \omega_i \circ \bar{\psi}_i \in \Phi. \tag{3.18}$$

This system satisfies the conditions (I)–(V) automatically. The same properties mentioned in the Appendix are derived for the energy, the upper bound of specific heat, correlation functions, order parameters and topology of phase diagram.

In contrast to general gauge symmetric models, more explicit relations can be derived in the present system. The partition function satisfies

$$\begin{aligned} Z(K;\boldsymbol{\omega}) &= \int \mathrm{d}\mu\{\phi\}\,\exp\{-K\tilde{\mathcal{H}}_0(\phi\circ\boldsymbol{\omega})\} \\ &= \int \mathrm{d}\mu\{\phi\}\,U_\omega \exp\{-K\tilde{\mathcal{H}}_0(\phi\circ\boldsymbol{\omega})\} \\ &= Z_0(K). \end{aligned} \tag{3.19}$$

Thus, the free energy and its higher derivatives in K are independent of K_p, and are identical with those in the pure system. This indicates that the thermal properties are identical with the pure system. At a glance, the present system is trivial and not worth studying. However, the response to the applied field such as

$$\mathcal{H}_{\mathrm{f}}(\phi) = -h\left(\sum_{i=1}^{N}\gamma(\phi_i) + \sum_{i=1}^{N}\gamma^\dagger(\phi_i)\right) \tag{3.20}$$

is non-trivial even at zero field. This can be seen in the behaviour of the correlation functions. Using the same technique as in (3.19), one derives

$$\begin{aligned}\langle\gamma(\bar{\phi}_0\circ\phi_r)\rangle_K &= \int \mathrm{d}\mu\{\phi\}\, U_\omega \frac{\exp\{-K\tilde{\mathcal{H}}_0(\phi\circ\omega)\}}{Z_0(K)}\gamma(\bar{\phi}_0\circ\phi_r)\\ &= \int \mathrm{d}\mu\{\phi\}\, \frac{\exp\{-K\tilde{\mathcal{H}}_0(\phi)\}}{Z_0(K)}\gamma(\bar{\phi}_0\circ\phi_r)\gamma^\dagger(\bar{\omega}_0\circ\omega_r)\\ &= f_0(r;K)\gamma^\dagger(\bar{\omega}_0\circ\omega_r)\end{aligned}\tag{3.21}$$

for any K, where

$$f_0(r;K)\equiv\int \mathrm{d}\mu\{\phi\}\,\frac{\exp\{-K\tilde{\mathcal{H}}_0(\phi)\}}{Z_0(K)}\gamma(\bar{\phi}_0\circ\phi_r)\tag{3.22}$$

is the correlation function in the pure system. Then, we get important relations;

$$f(r;K,K_p) = f_0(r;K)f_0^*(r;K_p)\tag{3.23}$$

$$g(r;K,K_p) = |f_0(r;K)|^2.\tag{3.24}$$

4. Phase diagrams and critical exponents

Equations (3.19), (3.23) and (3.24) provide relations for the phase diagram and critical exponents between the random and the pure systems. We discuss below three cases distinguished by the behaviours of the pure system.

4.1. Ferromagnetic pure systems

First, let us examine the case in which the pure system exhibits only the FM phase in low temperature region ($K = K_\mathrm{c}$); e.g the Ising model (3.6) in $d \geqslant 2$, the XY model (3.10) in $d \geqslant 3$ and so on. In sufficiently strong random regime, a freezing phase without FM long-range order appears besides the FM phase. We call this phase the spin-glass phase, since the behaviour of correlation function is identical; it is called the Mattis spin-glass phase in the Ising case [29, 30]. In this phase, the structure of the phase space and the dynamical behaviour which are important in real SG phenomena would be too simple and different from those investigated so far [1]. It is not our aim to apply the present theory directly to real SG problems. We just present models with SG-like behaviours in correlation functions.

We assume the asymptotic form of the correlation function in the pure system as

$$f_0(r;K)\sim\begin{cases}\dfrac{\exp\{-r/\xi_0(K)\}}{r^{d-2+\eta_0}} & (K<K_\mathrm{c})\\[2mm] \dfrac{1}{r^{d-2+\eta_0}} & (K=K_\mathrm{c})\\[2mm] m_0(K)^2+\dfrac{\exp\{-r/\xi_0(K)\}}{r^{d-2+\eta_0}} & (K>K_\mathrm{c}).\end{cases}\tag{4.1}$$

The critical exponents are defined by (4.1) and

$$m_0(K)\sim(K-K_\mathrm{c})^{\beta_0}\tag{4.2}$$

$$\xi_0(K)\sim|K-K_\mathrm{c}|^{-\nu_0}.\tag{4.3}$$

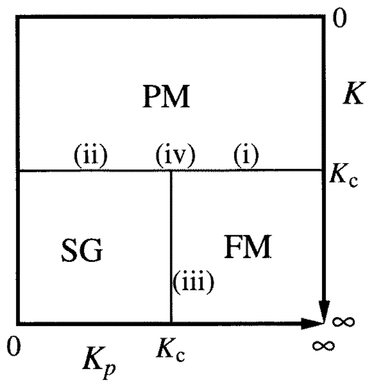

Figure 1. The phase diagram of the non-frustrated random model associated with an pure FM system. There are three kinds of phases, the PM, the FM and the SG. Four kinds of critical regimes are indicated as (i), (ii), (iii), and (iv).

Equations (3.23) and (3.24) with the definition of order parameters in section 2 lead to

$$m(K, K_p) = m_0(K)m_0(K_p) \tag{4.4}$$

$$q(K, K_p) = |m_0(K)|^2 \tag{4.5}$$

providing the phase diagram in figure 1. In such simple random systems, no re-entrant transition is observed. The result is quite similar to the asymmetric Mattis model [30]. The SG-like phase exhibiting the same behaviour of correlation functions in the SG phase always appears in systems associated with any FM pure systems in the present class of random spin systems. This is a common feature in gauge symmetric random systems [24].

Four kinds of critical regime exist in the phase diagram (see figure 1):

(i) $K \sim K_c$ and $K_p > K_c$,
(ii) $K \sim K_c$ and $K_p < K_c$,
(iii) $K > K_c$ and $K_p \sim K_c$,
(iv) $K \sim K_c$ and $K_p \sim K_c$ (the multicritical point).

Table 3. The relations of critical exponents between the pure and random systems. The critical regimes, (i), (ii), (iii) and (iv) in figure 1 are distinguished in the random system. The exponents, β, ν, η for the FM ordering and $\tilde{\beta}$, $\tilde{\nu}$, $\tilde{\eta}$ for the SG ordering are related to β_0, ν_0, η_0 in the pure system by use of the exact relations (3.18) and (3.19). The bars indicate that corresponding quantities are analytic and do not exhibit singularities in these regimes.

	(i)	(ii)	(iii)	(iv)
$\tilde{\beta}$	$2\beta_0$	$2\beta_0$	—	$2\beta_0$
$\tilde{\nu}$	ν_0	ν_0	—	ν_0
$\tilde{\eta}$	$d-2+2\eta_0$	$d-2+2\eta_0$	—	$d-2+2\eta_0$
β	β_0	—	β_0	$2\beta_0$
ν	ν_0	—	ν_0	ν_0
η	η_0	—	η_0	$d-2+2\eta_0$

The critical exponents in the random system are denoted by ν, β, η for the FM ordering and $\tilde{\nu}$, $\tilde{\beta}$, $\tilde{\eta}$ for the SG ordering. The asymptotic forms of $f(r; K, K_p)$ and $g(r; K, K_p)$ are assumed similarly as in (4.1) with $m_0(K)$, $\xi_0(K)$ and η_0 replaced by $m(K, K_p)$, $\xi(K, K_p)$ and η, and $q(K, K_p)$, $\tilde{\xi}(K, K_p)$ and $\tilde{\eta}$, respectively. Applying (3.23) and (3.24) to the above definitions, we obtain the relations of critical exponents between the pure and random systems. The results are summarized in table 3. As for the FM critical exponents, the universality (constantness of β and ν) and the weak universality (constantness of β/ν and η)

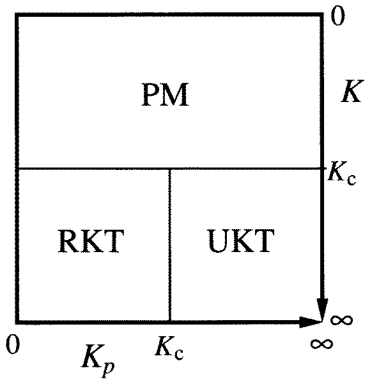

Figure 2. The phase diagram of the non-frustrated random model associated with a KT-type pure system. Three kinds of phases, the PM, the uniform KT (UKT) and the random KT (RKT), are indicated.

hold along the boundary of FM phase, (i) and (iii), except at the multicritical point. The universality and weak universality for the SG critical exponents hold along the boundary of PM phases, (i) and (ii), including the multicritical point.

4.2. Kosterlitz–Thouless-type pure systems

Next, we consider the case in which the pure system exhibits the KT phase [31] in temperature region below $K = K_c$; e.g. the XY model (3.10) in $d = 2$. Strictly speaking, the global symmetry mentioned in section 2 is not broken in the KT phase. One can define the KT phase by the region where the correlation length diverges. As discussed in the previous subsection, two different correlation lengths,

$$\xi(K, K_p) \equiv \overline{\lim_{r\to\infty}} \left| \frac{\partial}{\partial r} \log |f(r; K, K_p)| \right|^{-1} \tag{4.6}$$

$$\tilde{\xi}(K, K_p) \equiv \overline{\lim_{r\to\infty}} \left| \frac{\partial}{\partial r} \log g(r; K, K_p) \right|^{-1} \tag{4.7}$$

are defined in the corresponding random system. The phase diagram is obtained as in figure 2. There are two kinds of KT phases, the uniform KT (UKT) and the random KT (RKT) defined by

$\xi < +\infty$ and $\tilde{\xi} < +\infty$ in the PM phase,
$\xi = +\infty$ and $\tilde{\xi} = +\infty$ in the UKT phase,
$\xi < +\infty$ and $\tilde{\xi} = +\infty$ in the RKT phase.

In the UKT phase which is continued from the KT phase of the pure system, both the FM and the SG correlations are critical. In the RKT phase, the SG correlation is critical while the FM one is short-range. In the XY gauge glass model in two dimension, it has not yet been clarified if these KT phases exist or not. Korshunov discussed the instability of KT phases in any finite random regime [32]. The present theory provides an example of random systems which exhibits the UKT as well as the RKT phases in the low-temperature region.

4.3. Two-dimensional clock systems

Finally, we consider the case in which the pure system exhibits successive phase transitions, PM $\to$ KT $\to$ FM, like the q-state clock model (3.10) with $q \geqslant 5$ in two dimensions [33]. The PM–KT and KT–FM transitions are supposed to occur at $K = K_{c1}$ and $K = K_{c2}$,

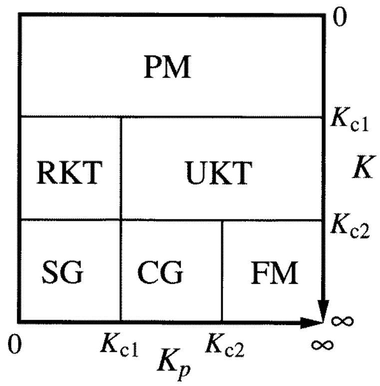

Figure 3. The phase diagram of the non-frustrated random model associated with the pure clock model in $d = 2$. There exist six kinds of phases, the PM, the FM the SG, the UKT, the RKT and the critical glass (CG).

Table 4. Behaviours of correlation functions in the phases shown in figure 3.

Phase	FM correlation	SG correlation
PM	Short range	Short range
FM	Long range	Long range
SG	Short range	Long range
UKT	Critical	Critical
RKT	Short range	Critical
CG	Critical	Long range

respectively. The phase diagram of the non-frustrated random system is obtained as in figure 3. There are six kinds of thermodynamic phases, the PM, the FM, the SG, the UKT, the RKT and the critical glass (CG) phases (see table 4). In the CG phase, the SG correlation is long-range while the FM one is critical. This means that the snap shot of the system looks like that in the KT phase while spins are freezing. Such behaviour has not been observed in real material.

5. Remarks

We propose a class of random spin systems without frustration. All models have gauge symmetry, and the method of gauge transformation can be applied to them. The phase diagram and critical exponents for both SG and FM orderings are related with those in the original pure system. The result for the absence of re-entrance is not completely rigorous in the method of gauge transformation. It can be shown exactly for the present models—especially for the XY gauge glass-like model with the Hamiltonian (3.10), however, one should note that the randomness is much different from the XY gauge glass model. They are regarded as rigorous examples for the results of the gauge transformation. They are also rigorous examples for the existence of the SG-like phases, in which the behaviour of the correlations is similar, with various symmetries.

The present theory is almost applicable to non-Abelian systems except for the argument of correlations. In non-Abelian systems, the system is gauge symmetric providing the same results in the appendix and exact relations such as (3.23) and (3.24) can be derived with small modification; order parameters become matrices. However, it is not straightforward to obtain the phase diagram exactly from such matrix relations.

Since the present models can be analysed exactly, they help us to understand the phase transition and the critical phenomena in random systems with controversial properties.

Appendix. Gauge symmetry and the method of gauge transformation

A system with the following conditions (I)–(V) is gauge symmetric and has the properties mentioned below [24].

(I) The set Φ forms a topological group with an operation denoted by $\phi \circ \psi$.

(II) The measure $\mathrm{d}\nu\{\boldsymbol{\omega}\}$ is an invariant measure;

$$\int \mathrm{d}\nu\{\boldsymbol{\omega}\}\, V_\psi \ldots = \int \mathrm{d}\nu\{\boldsymbol{\omega}\} \ldots . \tag{A1}$$

(III) The Hamiltonian is invariant in terms of the gauge transformation as

$$U_\psi V_\psi \mathcal{H}(\phi; \boldsymbol{\omega}) = \mathcal{H}(\phi; \boldsymbol{\omega}). \tag{A2}$$

(IV) The distribution function is transformed as

$$V_\psi P(\boldsymbol{\omega}; K_p) = Y(K_p)^{-1} \exp\{-K_p \tilde{\mathcal{H}}(\psi; \boldsymbol{\omega})\} \tag{A3}$$

providing

$$P(\boldsymbol{\omega}; K_p) = Y(K_p)^{-1} \exp\{-K_p \tilde{\mathcal{H}}(\phi_{\mathrm{E}}; \boldsymbol{\omega})\} \tag{A4}$$

where $\phi_{\mathrm{E}} = (\phi_{\mathrm{E}}, \phi_{\mathrm{E}}, \ldots, \phi_{\mathrm{E}})$ and

$$Y(K_p) = \int \mathrm{d}\nu\{\boldsymbol{\omega}\}\, \exp\{-K_p \tilde{\mathcal{H}}(\phi_{\mathrm{E}}; \boldsymbol{\omega})\}. \tag{A5}$$

(V) The Hamiltonian is invariant in terms of the global transformation;

$$\tilde{U}_\psi \mathcal{H}(\phi; \boldsymbol{\omega}) = \mathcal{H}(\phi; \boldsymbol{\omega}). \tag{A6}$$

In the system satisfying the above conditions, the energy can be obtained exactly on $K = K_p$

$$E(K_p, K_p) = -J \frac{\partial}{\partial K_p} \ln Y(K_p) \tag{A7}$$

and an upper bound on the specific heat is also obtained as

$$C(K_p, K_p) \leqslant k_{\mathrm{B}} K_p^2 \frac{\partial^2}{\partial K_p^2} \ln Y(K_p). \tag{A8}$$

The line $K = K_p$ is called Nishimori line [22] in the Ising case.

The FM correlation functions satisfies

$$f(r; K, K_p) \equiv [\langle \gamma(\bar{\phi}_0 \circ \phi_r)\rangle_K \langle \gamma^\dagger(\bar{\phi}_0 \circ \phi_r)\rangle_{K_p}]_c \tag{A9}$$

which provides the exact relation between the FM and the SG correlation functions on $K = K_p$;

$$f(r; K_p, K_p) = g(r; K_p, K_p). \tag{A10}$$

Using these relations, one obtains the following properties for the phase diagram [22–24]:

(a) There exits no SG phase on $K = K_p$.

(b) The boundary of FM phase is vertical or re-entrant.

(c) Using the modified model [23, 24, 34], a plausible argument for the verticality can be made.

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